

UMD seminar - Kevin Wilson

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A new description of the Bruhat-Tits group scheme

let $L = \text{field}$, complete wrt discrete valuation v

U1

$\mathcal{O} = \text{ring of integers}$

①

π = uniformizer in \mathcal{O}

$$k = \frac{\mathcal{O}}{\pi\mathcal{O}} \quad \text{residue field}, \quad k = \bar{k}.$$

let G be a split connected semisimple

group defined over \mathcal{O} .

$G \supset T = \text{split maximal torus}$

$B(G) = \text{building of } G(L).$

Fix an apartment $A = X_*(T) \otimes \mathbb{R}$

U1

base alcove a

ψ

origin 0.

①

Associated to any facet $\mathcal{F} \subset \mathcal{P}(G)$, one can define a parahoric Bruhat-Tits group scheme

$\mathcal{G}_{\mathcal{F}}$ over \mathcal{O} .

Defn: $\mathcal{G}_{\mathcal{F}}$ is the unique affine group scheme over \mathcal{O} s.t.

- (1) $\mathcal{G}_{\mathcal{F}}$ is smooth
 - (2) generic fiber $(\mathcal{G}_{\mathcal{F}})_L$ equals G_L .
 - (3) $\mathcal{G}_{\mathcal{F}}(\mathcal{O}) = \text{Fix}_{G(L)}(\mathcal{F}) \cap \ker(\chi_G)$
- ↑
Kottwitz homomorphism
parahoric subgroup associated to \mathcal{F} .

$$\chi_0 : G(L) \longrightarrow X^*(Z(\hat{G})).$$

This describes $\mathcal{G}_{\mathcal{F}}$ as a scheme. Can also

think of $\mathcal{G}_{\mathcal{F}}$ as a functor, i.e.

a map $\mathcal{G}_{\mathcal{F}} : \mathcal{O}\text{-algebras} \longrightarrow \text{Groups}$

(2)

This defn does not give a general ~~one~~ description of the functor of points.

Can give case by case ~~one~~ description in terms of lattice chains.

Recall: Let V be a finite rank projective \mathcal{O} -module

\Downarrow
finite rank free \mathcal{O} -module

Then an \mathcal{O} -lattice $\Lambda \subset V \otimes L$ is an \mathcal{O} -submodule which is finitely generated and contains an L -basis of $V \otimes L$. Thus, in particular,

$\Lambda = \bigoplus_{i=1}^r \mathcal{O} f_i$ where $\{f_i\}$ is a basis of $V \otimes L$.

Ex: $G = SL_3$, $M = \mathfrak{a}$. Let V = standard rep'n of G , standard basis $\langle e_1, e_2, e_3 \rangle$.

Consider the lattice chain

$$\mathcal{O}^3 \subset \mathcal{O} \oplus \mathcal{O}^2 \subseteq \mathcal{O}^2 \oplus \mathcal{O} \subset \mathcal{O}^3$$

↓

i.e. $\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{O}e_3$

We want to describe \mathcal{G}_γ using this lattice chain. Well, we should get the Invhor.

$$I = \begin{pmatrix} 0^\times & 0 & 0 \\ 0 & 0^\times & 0 \\ 0 & 0 & 0^\times \end{pmatrix}.$$

Then, I is exactly the subgroup of GL that stabilizes the lattice chain.

Then, we ~~get~~ have

$$\mathcal{G}_\gamma(R) = \left\{ (g_1, g_2, g_3) \in SL_3(0 \oplus \bar{1}\theta^2)(R) \times SL_3(0^2 \oplus \bar{1}\theta)(R) \times SL_3(0^3)(R) \text{ such that} \right.$$

$$\bar{1}\theta^3 \otimes R \rightarrow 0 \oplus \bar{1}\theta^2 \xrightarrow{(0^2 \oplus \bar{1}\theta) \otimes R} 0^3 \otimes R$$

$$g_3 \downarrow \qquad g_1 \downarrow \qquad g_2 \downarrow \qquad g_3 \downarrow$$

$$\bar{1}\theta^3 \otimes R \longrightarrow 0 \oplus \bar{1}\theta^2 \otimes R \xrightarrow{(0^2 \oplus \bar{1}\theta) \otimes R} 0^3 \otimes R$$

}

where $\gamma = \alpha = \text{base alcove}$

More generally: Let $\text{Rep}_{\mathcal{O}}(G)$ = category of

finite rank free \mathcal{O} -modules which carry a G -module structure.

Tannakian Duality (Saavedra)

$$G(R) = \{ (g_v^R) \in \prod_{V \in \text{Rep}_{\mathcal{O}}(G)} GL(V)(R) \text{ s.t. }$$

(*)

(a) If $V, W \in \text{Rep}_{\mathcal{O}}(G)$ and $\phi: V \rightarrow W$ a G -morphism, then

$$\begin{array}{ccc} V \otimes_{\mathcal{O}} R & \xrightarrow{\phi^R} & W \otimes_{\mathcal{O}} R \\ g_V^R \downarrow & & \downarrow g_W^R \\ V \otimes_{\mathcal{O}} R & \xrightarrow{\phi^R} & W \otimes_{\mathcal{O}} R \end{array} \quad \text{commutes}$$

$$(b) g_{\mathcal{O}}^R = \text{Id}_R$$

(c) If $V, W \in \text{Rep}_{\mathcal{O}}(G)$, ~~not~~ then

$$\begin{array}{ccc} (V \otimes_{\mathcal{O}} R) \otimes_R (W \otimes_{\mathcal{O}} R) & \xrightarrow{\sim} & (V \otimes_{\mathcal{O}} W) \otimes_{\mathcal{O}} R \\ \downarrow g_V^R \otimes g_W^R & & \downarrow g_{V \otimes W}^R \\ (V \otimes_{\mathcal{O}} R) \otimes (W \otimes_{\mathcal{O}} R) & \xrightarrow{\sim} & (V \otimes_{\mathcal{O}} W) \otimes_{\mathcal{O}} R \end{array} \quad \text{commutes}$$

$$(V \otimes_{\mathcal{O}} R) \otimes (W \otimes_{\mathcal{O}} R) \xrightarrow{\sim} (V \otimes_{\mathcal{O}} W) \otimes_{\mathcal{O}} R$$

(5)

How to get the lattice chain, given an arbitrary repr: Use Moy-Prasad filtration

Recall that the set of affine transformations

on $X^*(T) \otimes \mathbb{R}$ can be identified with

$$X^*(T) \otimes \mathbb{R} \times \mathbb{R} = A \times \mathbb{R}.$$

If $(x, r) \in A \times \mathbb{R}$, set

$$(x, r)(\lambda \otimes s) = r - \langle \lambda \otimes s, x \rangle.$$

Defn: If $V \in \text{Rep}_G(G)$ and $(x, r) \in A \times \mathbb{R}$,

then $V_{x,r} = \sum_{\lambda \in X^*(T), n \in \mathbb{Z}} V^\lambda \otimes \pi^n \circ \rho = \bigoplus_{\lambda} V^\lambda \otimes \pi^{\langle \lambda, r \rangle}$

$$n \geq (x, r)(\lambda)$$

where

$$\pi_{x,r} = \Gamma(x, r)(\lambda)$$

Note: If $r \geq s$, then $V_{x,r} \subseteq V_{x,s}$.

So for every (x, r) , we get a lattice

Ex: $G = SL_3$, $\mathfrak{g} = \mathfrak{a}$, want to recreate

$$\Pi\Theta^3 \subset \Theta \oplus \Pi\Theta^2 \subset \Theta^2 \oplus \Pi\Theta \subset \Theta^3$$

Want to be able to choose ~~some~~ some $x \in \mathfrak{a}$, $r \in \mathbb{R}$, to get the above lattice chain.

Here, the thing that works is

$$x = \frac{1}{4} (\sum \text{fundamental coroots}) = \frac{1}{4} (\alpha^\vee + \beta^\vee)$$

Then

$$V_{x, \frac{3}{4}} \subset V_{x, \frac{1}{4}} \subset V_{x, 0} \subset V_{x, -\frac{1}{4}}$$

is the lattice chain —

Defn: $\text{Aut}_{\mathfrak{g}}$ is a functor

$\text{Aut}_{\mathfrak{g}} : \Theta\text{-algebras} \longrightarrow \text{Groups}$

$$\text{Aut}_{\mathfrak{g}}(R) = \left\{ \left(g_{V_{x,r}}^R \right) \in \prod_{V \in \text{Rep}_0(G)} \text{GL}(V_{x,r})(R) \mid \begin{array}{l} x \in \mathfrak{g} \\ r \in \mathbb{R} \end{array} \right.$$

such that

(7)

(1) 4 conditions hold }

2 of the conditions are:

(1) $\forall (x, r), (y, s), \forall n \in \mathbb{N}$ s.t.

$V_{x, r+n} \subset V_{y, s} \subset V_{x, r}$, then

$$\underbrace{V_{x, r} \otimes_R V_{x, r+n}}_{\text{---}} \rightarrow V_{x, r+n}$$

$$\begin{array}{ccccccc} V_{x, r} \otimes_R R & \xrightarrow{\sim} & V_{x, r+n} \otimes_R R & \rightarrow & V_{y, s} \otimes_R R & \rightarrow & V_{x, r} \otimes_R R \\ g_{V_{x, r}}^R \downarrow & & g_{V_{x, r+n}}^R \downarrow & & g_{V_{y, s}}^R \downarrow & & g_{V_{x, r}}^R \downarrow \\ V_{x, r} \otimes_R R & \xrightarrow{\sim} & V_{x, r+n} \otimes_R R & \rightarrow & V_{y, s} \otimes_R R & \rightarrow & V_{x, r} \otimes_R R \end{array}$$

commutes.

(2) $\forall V, W \in \text{Rep}_R(G)$, $\forall (x, r), (y, s), (z, t)$

$$\text{s.t. } [V(x, r)(\lambda)] + [V(y, s)(\mu)] \geq [V(z, t)(\lambda \otimes \mu)]$$

where λ is a weight of V , μ a weight of W , then

$$(V_{x,r} \otimes_R R) \otimes (W_{y,s} \otimes_R R) \rightarrow (V \otimes W)_{z,t} \otimes_R R$$

$$g_{V_{x,r}}^R \otimes g_{W_{y,s}}^R \downarrow \qquad \qquad \qquad g_{(V \otimes W)_{z,t}}^R \downarrow$$

$$(V_{x,r} \otimes_R R) \otimes_R (W_{y,s} \otimes_R R) \rightarrow (V \otimes W)_{z,t} \otimes_R R$$

commutes.

In the ~~interest~~ interest of time, he didn't put the other 2 conditions

Thm (-) : ① Aut_γ is represented by an

affine group scheme over \mathcal{O} of finite type s.t.

$$- (\text{Aut}_\gamma)_L = G_L = (G_\gamma)_L$$

$$- \text{Aut}_\gamma(\mathcal{O}) = G_\gamma(\mathcal{O})$$

Moreover, $\exists!$ homomorphism of group schemes

$$\phi: G_\gamma \longrightarrow \text{Aut}_\gamma$$

which is ~~a~~ the identity on the generic fiber and ~~is~~ on \mathcal{O} -parts.

- ② If $\text{char}(k) = 0$, then ϕ is an isomorphism.